CENTER FOR CYBERNETICS STUDIES

The University of Texas
Austin, Texas 78712

31

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Center for Cybernetic Studies The University of Texas	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

An Explicit General Solution in Linear Fractional Programming

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Center for Cybernetic Studies - March, 1971

5. AUTHOR(S) (First name, middle initial, last name)

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6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March, 1971	26	22

8a. CONTRACT OR GRANT NO. NR 047-021 b. PROJECT NO. N00014-67-A-0126-0008 c. N00014-67-A-0126-0009 d.	9a. ORIGINATOR'S REPORT NUMBER(S) Center for Cybernetic Studies Research Report CS 58 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been released for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Not applicable.	Office of Naval Research (Code 434) Washington, D.C.

13. ABSTRACT

A complete analysis and explicit solution is presented for the problem of linear fractional programming with interval programming constraints whose matrix is of full row rank. The analysis proceeds by simple transformation to canonical form, exploitation of the Farkas-Minkowski lemma and the duality relationships which emerge from the Charnes-Cooper linear programming equivalent for general linear fractional programming. The formulations as well as the proofs and the transformations provided by our general linear fractional programming theory are here employed to provide a substantial simplification for this class of cases. The augmentation developing the explicit solution is presented, for clarity, in an algorithmic format.

Unclassified
Security Classification

Unclassified
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Linear fractional programming						
Linear programming						
Interval programming						
Algorithm						
Duality						
Dual problems						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Unclassified
Security Classification

Graduate School of Business

Research Report
CS 58

AN EXPLICIT GENERAL SOLUTION IN LINEAR FRACTIONAL PROGRAMMING

by

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July, 1971

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This research was partly supported by a grant from the Farah Foundation and by ONR Contracts N00014-67-A-0126-0008 and N00014-67-A-0126-0009 with the Center for Cybernetic Studies, The University of Texas. This report was also prepared as part of the activities of the Management Sciences Research Group at Carnegie-Mellon University under Contract N00014-67-A-0314-0007 NR 047-048 with the U.S. Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

We wish to thank W. Szwarc of the University of Wisconsin for comments which helped us to improve the exposition in this manuscript.

CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director

ABSTRACT

A complete analysis and explicit solution is presented for the problem of linear fractional programming with interval programming constraints whose matrix is of full row rank. The analysis proceeds by simple transformation to canonical form exploitation of the Farkas-Minkowski lemma and the duality relationships which emerge from the Charnes-Cooper linear programming equivalent for general linear fractional programming. The formulations as well as the proofs and the transformations provided by our general linear fractional programming thoery are here employed to provide a class of cases. The argumentation developing the explicit solution is presented, for clarity, in an algorithmic format.

I. Introduction:

The linear fractional programming problem arises in many contexts with relatively simple constraint sets -- e.g., in the reduction of integer programs to knapsack problems, in attrition games and in Markovian replacement problems as well as in Neyman-Pearson rejection region selection problems. Illustrative examples are provided by G. Bradley [5] or F. Glover and R. E. Woolsey [12][1/], J. Isbell and W. Marlow [13], C. Derman [10], and M. Klein [16].

The linear fractional programming problem in all generality, and with all singular cases considered, was reduced in [8] to at most a pair of ordinary linear programming problems. This immediately made available all of the algorithms, interpretations, etc., that are associated with linear programming. This includes, we should note, access to any ordered field,[2/] and any of the algorithms and the computer codes for linear programming problems which, by virtue of [8], thereby also become available for any problem in linear fractional form. Thus, with the development in [8], the work in linear fractional programming took a different form from its previous sole concern with the development of special types of algorithms for dealing with this kind of problem.

In the present paper, we apply our reduction, as given in [8], to a general class of linear fractional problems -- viz., those for which the constraint set is given by

(1.1) $\quad a \leq A\,x \leq b$

so that this part of the model is in "interval programming" form.[3/]

1/ See also E. Balas and M. Padberg [1].

2/ See, e.g., the development of the opposite sign theorem and related developments in [7].

3/ See [2] - [4] and [18]-[19].

Here we shall assume that the matrix A is full row rank and the vectors a, b and x meet the usual conditions for conformance. This means that the constraint set is a parallelopiped. See the Final Appendix in [7].

Subject to conditions (1.1), we wish to

$$\text{maximize } R(x) \equiv \frac{N(x)}{D(x)} \equiv \frac{c^T x + c_o}{d^T x + d_o} \not\equiv \text{constant} \tag{1.2}$$

so that we are now concerned with a problem of linear fractional programming. Because A is of full row rank it has a right inverse, $A^{\#}$, and hence we can write

$$AA^{\#} = I. \tag{1.3}$$

Now, setting

$$y = Ax$$

or

$$x = A^{\#}y + Pz$$

(1.4) where

$$P \equiv I - A^{\#}A$$

and

$$z \text{ is arbitrary}$$

we obtain

$$\max R(y,z) = \frac{c^T A^{\#} y + c^T Pz + c_o}{d^T A^{\#} y + d^T Pz + d_o}$$

(1.5) Subject to

$$a \leq y \leq b$$

in place of (1.1) and (1.2).

Because z is arbitrary,[1/] unless

$$c^T P = d^T P = 0 \tag{1.6}$$

1/ Observe that we have ruled out the case in which R is identically constant in (1.2).

we shall obtain $\max R = \infty$. In order to avoid repetitious arguments, however, we defer the proof of this until we have discussed the situation $c^T P = d^T P = 0$. See section IV, below.

Waving this consideration aside, we shall next proceed to solve this problem explicitly and in all generality by means of the following 3 characterizations: The denominator $D(x)$ is (1) bisignant, (2) unisignant and non-vanishing, or (3) unisignant and vanishing on the constraint set. In (1), i.e., the bisignant case, we shall show that $R(x) = \infty$. Furthermore, we shall show how to identify this case at the outset so that it may be discarded from further consideration. This will leave us with only cases (2) and (3) to examine where we shall proceed to transformations from which a one-pass numerical comparison of coefficients makes explicit the optimal value and solution.

After this has all been done, we shall then return to assumption (1.6) in a way that utilizes the preceding developments. Finally we shall supply numerical examples to illustrate some of these situations and then we shall draw some conclusions for further research which will return to the remarks at the opening of this section.

II. Bisignant Denominators:

Employing assumption (1.6), our problem is

$$\text{(2.1)} \qquad \max R(y) = \frac{c^T A^{\#} y + c_o}{d^T A^{\#} y + d_o}$$

subject to

$$a \leq y \leq b$$

in place of (1.5). Note, however, that here, and in the following, we shall slightly abuse notation by continuing to use the symbols R, N, D, as in (1.1) and (1.2) even though we mean the transformed function, as in (2.1).

Let $\check{D}$, $\hat{D}$ denote the maximum and minimum, respectively, of D over the constraint set. We note:

Lemma 1:

(a) D is bisignant if and only if $\check{D} > 0$ and $\hat{D} < 0$.

(b) D is unisignant if and only if either $\check{D} \leq 0$ or $\hat{D} \geq 0$.

In terms of y, since we can choose each component of y independently -- see (1.5) -- we can express $\check{D}$ and $\hat{D}$ immediately as

$$(2.2)\qquad \begin{aligned} \check{D} &= \sum_{+} d'_j\, b_j + \sum_{-} d'_j\, a_j + d_o \equiv \max D(y) \\ \hat{D} &= \sum_{+} d'_j\, a_j + \sum_{-} d'_j\, b_j + d_o \equiv \min D(y) \end{aligned}$$

where $d'_j \equiv (d^T A^{\#})_j$ is the $j^{\underline{th}}$ element of $d^T A^{\#}$ and "+" or "−" indicates that the summation is on only the positive or negative d'_j.

Let us first consider the bisignant situation. If we make the transformation of variables

$$(2.3)\qquad \begin{aligned} y_j - a_j &= k_j \zeta_j,\ d_j \geq 0 \\ b_j - y_j &= k_j \zeta_j,\ d_j < 0 \end{aligned}$$

where the $k_j > 0$ will be suitably chosen, the constraints transform to

$$(2.4)\qquad \begin{aligned} &0 \leq k_j \zeta_j \leq b_j - a_j \\ \text{or}\quad &0 \leq \zeta_j \leq \delta_j \equiv \frac{b_j - a_j}{k_j}\ . \end{aligned}$$

Without loss of generality, the δ_j are positive, since otherwise $\zeta_j = 0$ and does not enter into the optimization.

By choosing the k_j suitably, we obtain the form

$$(2.5)\qquad R(\zeta) = \frac{\sum_j \gamma_j\, \zeta_j + \gamma_o}{\sum_{+} \zeta_j + \sum_{-} \zeta_j - 1}$$

Note that $\sum_{+} \delta_j + \sum_{-} \delta_j > 1$, since otherwise D would not be bisignant.

One of the following two cases must now hold:

Case (i) for some $\bar{\zeta}$, $0 \leq \bar{\zeta} \leq \delta$, such that $D(\bar{\zeta}) = 0$ we have $N(\bar{\zeta}) \neq 0$,

or else

Case (ii) for every ζ, $0 \leq \zeta \leq \delta$, such that $D(\zeta) = 0$ we have $N(\zeta) = 0$.

In case (i), since $N(\zeta)$ is continuous, there is a neighborhood of $\bar{\zeta}$ in which $N(\zeta)$ is unisignant. Since

$$\sum_{+} \bar{\zeta}_j + \sum_{-} \bar{\zeta}_j = 1 < \sum_{+} \delta_j + \sum_{-} \delta_j \tag{2.6}$$

and $0 \leq \zeta_j \leq \delta_j$ in the constraint set, we can choose $\epsilon_j \geq 0$, $\sum_j \epsilon_j > 0$, so that $0 \leq \bar{\zeta} \pm \epsilon \leq \delta$, $\operatorname{sgn} N(\bar{\zeta} + \epsilon) = \operatorname{sgn} N(\bar{\zeta} - \epsilon)$ and $D(\zeta + \epsilon) > 0 > D(\bar{\zeta} - \epsilon)$. By approaching $\bar{\zeta}$ along the line segment from one of $\bar{\zeta} + \epsilon$. $\bar{\zeta} - \epsilon$, we can make $R(\zeta) \to \infty$.

In case (ii), we must have $\gamma_j = 0$ for all j such that $d_j = 0$. For $D(\zeta) = 0$ involves specifying only the ζ_j for $d_j > 0$ and $d_j < 0$. If $\gamma_{j_o} \neq 0$ for $d_{j_o} = 0$, then having made $D(\zeta) = 0$, we can change the value of $N(\zeta)$ by changing ζ_{j_o}. Thus $D(\zeta) = 0$ would <u>not</u> imply $N(\zeta) = 0$. We therefore drop the "+", "—" notation in considering case (ii) and rewrite it as: $\sum_j \gamma_j \zeta_j + \gamma_o = 0$ whenever $0 \leq \zeta_j \leq \delta_j$ and $\sum_j \zeta_j = 1$.

Letting $y_j = \zeta_j y_o$, $y_o \geq 0$, this becomes,

$\pm (\sum_j \gamma_j y_j + \gamma_o y_o) \geq 0$ whenever

$$\begin{aligned} \sum_j y_j - y_o &= 0 \\ - y_j + \delta_j y_o &\geq 0 \\ y_j &\geq 0 \\ y_o &\geq 0 \end{aligned} \tag{2.7}$$

Note that the implication extends to $y_o = 0$ since $y_o = 0$ implies <u>all</u> $y_j = 0$.

We now apply the Farkas-Minkowski lemma[1/] to the pair of implications in (2.7) and obtain

$$(2.8)\qquad \begin{cases} \gamma_j = \mu^+ - \theta_j^+ + \nu_j^+ \\ \gamma_o = -\mu^+ + \sum_j \delta_j \; \theta_j^+ + \nu_o^+ \; ; \quad \theta_j^+, \; \nu_j^+, \; \nu_o^+ \geq 0 \end{cases}$$

for the first implication--<u>viz</u>., $(\sum_j \gamma_j \; y_j + \gamma_o \; y_o) \geq 0$. For the second one <u>viz</u>., $- (\sum_j \gamma_j \; y_j + \gamma_o \; y_o) \geq 0$, we obtain,

$$(2.9)\qquad \begin{cases} -\gamma_j = \mu^- - \theta_j^- + \nu_j^- \\ -\gamma_o = -\mu^- + \sum_j \delta_j \theta_j^- + \nu_o^- \; ; \quad \theta_j^-, \; \nu_j^-, \; \nu_o^- \geq 0 \end{cases}$$

Adding the first expressions in (2.8) and (2.9),

$$0 = \mu^+ + \mu^- - (\theta_j^+ + \theta_j^-) + (\nu_j^+ + \nu_j^-)$$

(2.10) or

$$\theta_j^+ + \theta_j^- = (\mu^+ + \mu^-) + (\nu_j^+ + \nu_j^-)$$

Adding the second pair

$$0 = - (\mu^+ + \mu^-) + \sum_j \delta_j \; (\theta_j^+ + \theta_j^-) \; (\nu_o^+ + \nu_o^-)$$

(2.11) or

$$\mu^+ + \mu^- = \sum_j \delta_j \; (\theta_j^+ + \theta_j^-) + \nu_o^+ + \nu_o^-$$

Since each term on the right is non-negative, we have

$$(2.12)\qquad \mu^+ + \mu^- \geq 0.$$

Next, substituting from (2.10) into (2.11), we get

$$0 = -(\mu^+ + \mu^-) + \sum_j \delta_j \; (\mu^+ + \mu^-) + \sum_j \delta_j \; (\nu_j^+ + \nu_j^-) + \nu_o^+ + \nu_o^-$$

(2.13) or

$$0 = (\sum_j \delta_j - 1) \; (\mu^+ + \mu^-) + \sum_j \delta_j \; \nu_j^+ + \sum_j \delta_j \; \nu_j^- + \nu_o^+ + \nu_o^- .$$

1/ See Appendix C in [7]

Since the right-hand side is a sum of non-negative terms, each of these must be zero. Moreover,

$$\begin{aligned}
&\mu^{+} + \mu^{-} = 0 \qquad \text{since } \sum_j \delta_j - 1 > 0 \\
(2.14) \qquad &\nu_j^{+} = \nu_j^{-} = 0 \qquad \text{since } \delta_j > 0 \\
&\nu_o^{+} = \nu_o^{-} = 0
\end{aligned}$$

By virtue of (2.14), and going back to (2.10),

$$(2.15) \qquad \theta_j^{+} + \theta_j^{-} = 0.$$

Further, with $\theta_j^{+}, \theta_j^{-} \geq 0$ we must have $\theta_j^{+} = \theta_j^{-} = 0$ for all j.

Therefore, $\gamma_j = \mu^{+}$, for all j, and $\gamma_o = -\mu^{+}$, so that we have

$$(2.16) \qquad \begin{aligned} R(\zeta) &= \frac{\sum_j \gamma_j \zeta_j + \gamma_o}{\sum_j \zeta_j - 1} = \frac{\mu^{+}(\sum_j \zeta_j - 1)}{\sum_j \zeta_j - 1} \\ &= \mu^{+} = \text{constant.} \end{aligned}$$

In other words, case (ii) can only occur in the trivial instance where the numerator is a constant multiple of the denominator. In this case, each coefficient in the numerator is the same multiple of the corresponding coefficient in the denominator, and this would have to be true in the original N(x), D(x) description and hence obvious upon comparing the initial coefficients. Since we have ruled out this very obvious case--see (1.2)--we have only max R (x) = ∞ when D (x) is bisignant on the constraint set.

III. Unisignant Denominators:

The unisignant cases now remain to be considered. If $\check{D} \leq 0$ we multiply both N and D by -1, (thus not altering the value of R) and we are then

reduced to "D" $\geq$ 0. With this normalization, we make a transformation of variables as in (2.3),

$$(3.1) \qquad \begin{aligned} y_j - a_j &= g_j \xi_j \ , \quad d_j \geq 0 \\ b_j - y_j &= g_j \xi_j \ , \quad d_j < 0 \end{aligned}$$

where the $g_j > 0$ will be suitably chosen. The constraint set will now be

$$(3.2) \qquad 0 \leq \xi_j \leq \delta_j = \frac{b_j - a_j}{g_j}$$

and, first considering the case where $\hat{D} > 0$, the g_j can be chosen so that the problem is

$$(3.3) \qquad \max R(\xi) = \frac{\sum_j \gamma_j \xi_j + \gamma_o}{\sum_j \xi_j + 1} \ , \quad 0 \leq \xi_j \leq \delta_j$$

In (3.3) the summation is only over "+" and "−" because, the denominator being positive, optimal values for the ξ_j such that $d_j = 0$ can be specified as $\xi_j = 0$ when $\gamma_j < 0$; $\xi_j = \delta_j$ when $\gamma_j \geq 0$, and these new constant terms are assumed to be already contained in γ_o. By the reduction that we gave in [8], however, the equivalent linear programming problem is

$$\max \ \sum_j \gamma_j \eta_j + \gamma_o \eta_o$$

subject to

$$(3.4) \qquad \begin{aligned} &\sum_j \eta_j + \eta_o = 1 \\ &\eta_j - \delta_j \eta_o \leq 0 \\ &\eta_j \geq 0 \ , \end{aligned}$$

where we can also note that these constraints imply $\eta_o > 0$.

The dual to (3.4) is

$$\min \quad u$$

subject to

$$(3.5) \qquad \begin{aligned} u + \omega_j &\geq \gamma_j \\ u - \sum_j \delta_j \omega_j &= \gamma_o \\ \omega_j &\geq 0 \end{aligned}$$

We shall employ this dual in an essential manner to obtain our desired one-pass argument for obtaining an optimum. At each step in the procedure, we shall have a solution to a less restrictive problem than the dual problem and an associated primal feasible solution.

Suppose the γ's are renumbered so that $\gamma_1 \geq \gamma_2 \geq \cdots \gamma_n$. The case (i) $\gamma_o \geq \gamma_1$ has the immediately obvious primal solution $\eta_o^* = 1$, $\eta_j^* = 0$ and $\max R(\xi) = \gamma_o$.

In the contrary case,

(ii) $$\gamma_1 \geq \cdots \geq \gamma_p > \gamma_o \geq \gamma_{p+1} \geq \cdots \geq \gamma_n ,$$

we build up an algorithm based on the dual problem in which we choose u^q at the $q^{\underline{th}}$ step to satisfy

$$(3.6) \qquad \begin{aligned} u^q + \omega_j^q &= \gamma_j, \quad j = 1,\ldots, q \\ u^q - \sum_{j=1}^{q} \delta_j \omega_j^q &= \gamma_o . \end{aligned}$$

Using the first q equations to obtain ω_j^q in terms of u^q and substituting in the last equation, we obtain

$$(3.7) \qquad u^q \left(1 + \sum_1^q \delta_j\right) = \sum_1^q \gamma_j \delta_j + \gamma_o$$

and hence

(3.8) $$u^q = \frac{\sum_{1}^{q} \gamma_j \delta_j + \gamma_o}{1 + \sum_{1}^{q} \delta_j}$$

Thus, u^q is a convex combination of $\gamma_o, \gamma_1, \ldots, \gamma_q$ with proportionality constants $1, \delta_1, \ldots, \delta_q$. Note that if $u^q \leq \gamma_q$ then u^q, ω_j^q, $j = 1, \ldots, q$, satisfy the first q constraints plus the "γ_o" constraint of the dual problem; hence satisfy a less restrictive problem than the dual.

Taking

(3.9) $$\eta_o^q = 1/(1 + \sum_{1}^{q} \delta_j)$$

and

(3.10) $$\eta_j^q = \delta_j/(1 + \sum_{1}^{q} \delta_j),\ j = 1, \ldots, q$$

$$\eta_j^q = 0, \quad j > q ,$$

then η_j^q, $j = 0, \ldots, n$, is a feasible solution to the primal problem and $\gamma^T\eta^q + \gamma_o\eta_o^q = u^q$. (By substitution in (3.4) and comparison with (3.8).)

Hence, <u>whenever we can get</u> u^q, ω_j^q <u>feasible for the dual problem</u>, <u>we will have a primal feasible solution</u> for η^q <u>with the same functional value and thus we will have an optimal pair of dual solutions</u>. This, plus the equivalences maintained via (3.1) and our theory from [ ], thus justifies the development that we now detail as follows:

To start,

(3.11) $$u^1 = (\gamma_o + \gamma_1\delta_1) \big/ (1 + \delta_1) .$$

(Note, $u^1 > \gamma_o$ since $\gamma_1 > \gamma_o$ and u^1 is a proper convex combination of γ_1, γ_o.)

We check:

Is $u^1 \geq \gamma_2$?

If yes:

we are done

$$u^1 = u^*$$

$$\omega_j^1 = \omega_j^* = \begin{cases} \gamma_j - u^*, & j = 1 \\ 0, & j > 1 \end{cases}$$

and

$$\eta_o^1 = 1/1+\delta_1$$

$$\eta_1^1 = \eta_1^* = \delta_1 / 1+\delta_1$$

$$\eta_j^1 = \eta_j^* = 0, \ j > 1.$$

If No:

$u^1 < \gamma_2$, then

$$u^2 = \frac{\gamma_o + \gamma_1\delta_1 + \gamma_2\delta_2}{1 + \delta_1 + \delta_2}$$

$$= \left(\frac{\gamma_o + \gamma_1\delta_1}{1 + \delta_1}\right)\left(\frac{1 + \delta_1}{1 + \delta_1 + \delta_2}\right) + \left(\frac{\delta_2}{1 + \delta_1 + \delta_2}\right)\gamma_2$$

$$= u^1\left(\frac{1 + \delta_1}{1 + \delta_1 + \delta_2}\right) + \left(\frac{\delta_2}{1 + \delta_1 + \delta_2}\right)\gamma_2 < \gamma_2,$$

since $u^1 < \gamma_2$.

Next, Is $u^2 \geq \gamma_3$?

If Yes: We are done with the substitutions indicated by (3.6) ff.

If No:

$u^2 < \gamma_3$ and we continue to u^3 .

This process must stop by u^p at the latest since

$$u^p = \frac{\gamma_o + \sum_1^p \gamma_j \delta_j}{1 + \sum_1^p \delta_j} > \gamma_o \geq \gamma_{p+1} \geq \cdots \geq \gamma_n.$$

Thus $\max R(\xi) = u^s$ where s is the least positive integer such that $u^s \geq \gamma_{s+1}$, and

$$u^s = \frac{\gamma_o + \sum_1^s \gamma_j \delta_j}{1 + \sum_1^s \delta_j}$$

This concludes the case $\hat{D} > 0$.

The remaining case has $\hat{D} = 0$. By making a transformation of variables as in (3.1), and choosing the $g_j > 0$ suitably, we obtain

$$(3.12) \qquad R(\xi) \equiv \frac{\sum_1^n \gamma_j \xi_j + \gamma_o}{\sum_1^n \xi_j}, \quad 0 \leq \xi_j \leq \delta_j .$$

We may dispose of two situations immediately

(i) $\gamma_o > 0$: then $R(\xi) \to \infty$ as $\xi_j \to 0$.

(ii) $\gamma_o = 0$: here the dual problems are

$$\begin{array}{llll} \max & \sum_1^n \gamma_j \eta_j & \min & u \\ \text{with} & & \text{with} & \\ & \sum_1^n \eta_j = 1 & & u + \omega_j \geq \gamma_j \\ & \eta_j - \delta_j \eta_o \leq 0 & & -\sum_1^n \delta_j \omega_j \geq 0 \\ & \eta_j, \eta_o \geq 0 & & \omega_j \geq 0 \end{array}$$

An optimal solution pair is $\omega_j^* = 0$, $u = \gamma_1 \equiv \max \gamma_j$ and $\eta_1^* = 1$, $\eta_j^* = 0$, $j \geq 2$. The maximum of $R(\xi)$ is thus γ_1.

The remaining instance is

(iii) $\quad \gamma_o < 0: \quad \gamma_1 \geq \cdots \geq \gamma_p \geq \gamma_o \geq \gamma_{p+1} \geq \cdots \geq \gamma_n.$

The dual linear programs can now be written

$$\begin{array}{llll} \max & \sum_j \gamma_j \eta_j + \gamma_o \eta_o & \qquad \min u & \\ \text{with} & \sum_j \eta_j = 1 & \qquad \text{with } u + \omega_j \geq \gamma_j & \\ & \eta_j - \delta_j \eta_o \leq 0 & \qquad \sum_j \delta_j \omega_j = |\gamma_o| & \\ & \eta_j \geq 0 & \qquad \omega_j \geq 0 & \end{array}$$

As before, we define

$$u^q + \omega_j^q = \gamma_j, \quad j = 1, \ldots, q$$

$$\sum_1^q \delta_j \omega_j^q = |\gamma_o| .$$

This yields

$$u^q = \left(\sum_1^q \gamma_j \delta_j + \gamma_o\right) / \sum_1^q \delta_j .$$

It may be easily verified that

$$u^q = u^{(q-1)} \left(\frac{\sum_1^{q-1} \delta_j}{\sum_j^q \delta_j} \right) + \left(\frac{\delta_j}{\sum_1^q \delta_j} \right) \gamma_q ,$$

or, what is the same thing, u^q is a proper convex combination of $u^{(q-1)}$ and u^q. Thus, if $u^r < \gamma_{r+1}$, than $u^r < u^{r+1} < \gamma_{r+1}$ as in our earlier argument for $\hat{D} > 0$.

The steps of our process are as before: if $u^q \geq \gamma_{q+1}$ we are done; otherwise, we test u^{q+1} against γ_{q+2}. At worst we are done with

$$u^n = (\sum_1^n \gamma_j \delta_j + \gamma_o) / \sum_1^n \delta_j .$$

As before, $u^s = \max R(\xi)$ where s is the least positive integer such that $u^s \geq \gamma_{s+1}$.

IV. R (y,z):

Returning to (1.5) we consider the remaining cases in which either

(a) $d^T P = 0, \qquad c^T P \neq 0$

(4) or

(b) $d^T P \neq 0.$

In case (a), since z is arbitrary we can make $c^T P\, z = | c^T P\, z | \to \infty$, hence $\max R(y,z) \to \infty$. In case (b), we are in the bisignant denominator situation since we can make $d^T P\, z \to \pm\infty$. The argument of the bisignant section of this paper (with the additional variables, z) now shows that max $R = \infty$ since we have ruled out, a priori, the case in which $R \equiv$ constant.

V. Examples:

Some examples may help to fix and sharpen some of the preceding developments. Thus consider

$$\max R(x) \equiv \frac{3\, x_1 - x_3 + 4}{2\, x_2}$$

(5.1) with

$$-1 \leq x_1 + x_3 \leq 2$$
$$1 \leq \quad x_2 \quad \leq 5.$$

and the variables x_1, x_2, x_3 are otherwise unrestricted.

Here we have, [1/]

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^{\#} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$P \equiv (I - A^{\#} A) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.2}$$

$$a = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

To exhibit the development in full detail we next write

$$c^T A^{\#} y = (3,0,-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (3,0) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 3y_1$$

$$c^T P z = (3,0,-1) \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = (0,0,-3) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = -3z_3$$

$$c_o = 4$$

(5.3)

$$d^T A^{\#} y = (0,2,0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (0,2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2\, y_2$$

$$d^T P z = (0,2,0) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = (0,0,0,) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = 0$$

$$d_o = 0$$

1/ It may be observed that the $A^{\#}$ choice is not unique.

Evidently in this case $d^T P = 0$, as witness the next to the last expression. On the other hand, $c^T P \neq 0$, as witness $c^T Pz = -3z_3$ in the second expression for (5.3). Hence condition (a) of the preceding section obtains and we have $R \to \infty$ even though

$$(5.4) \qquad \begin{aligned} -1 &\leq y_1 \leq 2 \\ 1 &\leq y_2 \leq 5. \end{aligned}$$

This occurs because z is arbitrary and can be freely chosen in

$$(5.5) \qquad \max R(y,z) \equiv \frac{c^T A^{\#} y + c^T P z + c_o}{d^T A^{\#} y + d^T P z + d_o} = \frac{3y_1 - 3z_3 + 4}{2y_2}$$

which is the specialization of (1.5) to this case. Of course, the result $R \to \infty$ in (5.1) can be confirmed by direct inspection, since negative values of x_3 may be selected along with increasingly positive values of x_1 as required in order to maintain the first interval programming constraint.

This last remark suggests that an adjunction such as

$$(5.6) \qquad 0 \leq x_3 \leq 1$$

will convert (5.1) to a problem with a finite maximum. This yields an A for an interval programming format with the full row rank condition fulfilled as in

$$(5.7) \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^{\#} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On the other hand, A is also of full column rank so that we also have $A^{\#} = A^{-1}$ and

$$P = I - A^{\#} A = 0.$$

Hence both of the conditions specified in (1.6) are fulfilled -- viz.

$$c^T P = d^T P = 0$$

for any c^T and d^T.

The problem to be solved is now written

$$\max R(y) \equiv \frac{c^T A^{\#} y + c_o}{d^T A^{\#} y + d_o} = \frac{3y_1 - 4y_3 + 4}{2 y_2}$$

(5.8) with

$$-1 \leq y_1 \leq 2$$

$$1 \leq y_2 \leq 5$$

$$0 \leq y_3 \leq 1$$

Evidently the solution to this problem is $y_1^* = 2$, $y_3^* = 0$ and $y_2^* = 1$ so that

$$\max R(y) = \frac{10}{2} = 5 \cdot$$

To obtain the corresponding components of x we simply utilize (1.4) with $P = 0$ to obtain

$$(5.9) \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{\#} y = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

As may be seen, these x values satisfy (5.1) with (5.6) adjoined. They are evidently also maximal with $R(x) = 5$ since x_3 can no longer be negative and x_2 and x_1 are at their lower and upper limits, respectively.

In some cases the solutions, as above, may be obvious but, of course, this cannot always be expected. Recourse to the preceding development, however, will produce the wanted results in any case, however, as we illustrate by now developing the above examples, along with the related background materials, in some detail as follows:

Because the denominator is unisignant we utilize section III. Observing that $d_j = d_2 = 2$ in the denominator and hence is non-negative, we have recourse only to the first part of (3.1) on page 8 in order to write

$$
\begin{aligned}
y_1 - a_1 &= g_1 \xi_1 \\
y_2 - a_2 &= g_2 \xi_2 \qquad (6.1)\\
y_3 - a_3 &= g_3 \xi_3
\end{aligned}
$$

where, respectively, $a_1 = -1$, $a_2 = 1$ and $a_3 = 0$, via (5.8). The development from (3.1) to (3.2) on page 8 applies to this case as,

with

$$
\begin{aligned}
0 \le \xi_1 \le \delta_1 &= \frac{b_1 - a_1}{g_1} = \frac{2 - (-1)}{g_1} \\
0 \le \xi_2 \le \delta_2 &= \frac{b_2 - a_2}{g_2} = \frac{5 - 1}{g_2} \qquad (6.2)\\
0 \le \xi_3 \le \delta_3 &= \frac{b_3 - a_3}{g_3} = \frac{1 - 0}{g_3} \quad .
\end{aligned}
$$

The insertion of (6.1) into the functional then produces

$$
\begin{aligned}
&\frac{3\,(g_1 \xi_1 + a_1) - 4\,(g_3 \xi_3 + a_3) + 4}{2\,(g_2 \xi_2 + a_2)} = \qquad (6.3)\\
&= \frac{3\, g_1 \xi_1 - 4\, g_3 \xi_3 + 1}{2\,(g_2 \xi_2 + 1)}
\end{aligned}
$$

via (5.8). Choosing

$$ g_1 = 1/2, \quad g_2 = 1, \quad g_3 = 1/2 \qquad (6.4) $$

and setting

$$ \gamma_1 = 3/2, \quad \gamma_2 = 0, \quad \gamma_3 = -4/2, \quad \gamma_o = 1/2 \qquad (6.5) $$

gives the denominator form wanted for (3.3) as:

$$\max R(\xi) = \frac{\frac{3}{2}\xi_1 - \frac{4}{2}\xi_3 + \frac{1}{2}}{\xi_2 + 1}$$

(6.6) with

$$0 \le \xi_1 \le 6$$
$$0 \le \xi_2 \le 4$$
$$0 \le \xi_3 \le 2$$

The transformation $\xi_j = \eta_j/\eta_o$ from our previously developed theorem [8] then produces the following example for (3.4):

$$\max \ \frac{3}{2}\eta_1 + 0\eta_2 - 2\eta_3 + \frac{\eta_o}{2}$$

with

(6.7)

$$\begin{aligned} \eta_1 + \eta_2 + \eta_3 + \eta_o &= 1 \\ \eta_1 \qquad\qquad -6\eta_o &\le 0 \\ \eta_2 \qquad -4\eta_o &\le 0 \\ \eta_3 -2\eta_o &\le 0 \\ \eta_1, \eta_2, \eta_3 &\ge 0 \end{aligned}$$

where the g_j values of (6.4) combine with (6.2) to give $\delta_1 = 6$, $\delta_2 = 4$ and $\delta_3 = 2$, as required for the application of (3.4). The corresponding dual, which our previous theory also gives access to, is

$$\min \ u$$

with

(6.8)

$$\begin{aligned} u + \omega_1 \qquad\qquad &\ge \frac{3}{2} \\ u + \qquad \omega_2 \qquad &\ge 0 \\ u + \qquad\qquad \omega_3 &\ge -2 \\ u - 6\omega_1 - 4\omega_2 - 2\omega_3 &= \frac{1}{2} \\ \omega_1, \omega_2, \omega_3 &\ge 0. \end{aligned}$$

This, of course, is the application of (3.5) to the present example.

Since $\gamma_1 = \frac{3}{2}$ exceeds $\gamma_o = \frac{1}{2}$ we are in the situation of case (ii) following (3.5). Thus, preserving our subscript identifications from (6.5), we have

$$\gamma_1 > \gamma_o \geq \gamma_2 \geq \gamma_3 \tag{6.9}$$

in our present situation. We therefore see that the first application of the suggested algorithm should suffice. (See the remarks which conclude the case $\hat{D} > 0$ in section III).

Applying (3.11) now produces

$$u^1 = \left(\frac{1}{2} + \frac{3}{2}\cdot 6\right)\Big/(1 + 6) = \frac{19}{14} > \gamma_o = \frac{1}{2} \tag{6.10}$$

Evidently this also formally satisfies the condition that u^1 equals or exceeds the immediate successor of γ_1 in (6.9). Hence, we have

$$\begin{aligned} u^1 &= u^* = \frac{19}{14} \\ \omega_1^1 &= \omega_1^* = \gamma_1 - u^* = \frac{3}{2} - \frac{19}{14} = \frac{2}{14} \\ \omega_2^1 &= \omega_2^* = 0 \\ \omega_3^1 &= \omega_3^* = 0 \end{aligned} \tag{6.11}$$

which satisfy the constraints of (6.8), as may be verified, with $\min u = u^* = 19/14$.

Moving to the primal problem via (3.9),

$$\begin{aligned} \eta_o^1 &= \frac{1}{1+\delta_1} = \frac{1}{1+6} = \frac{1}{7} \\ \eta_1^1 &= \frac{\delta_1}{1+\delta_1} = \frac{6}{1+6} = \frac{6}{7} \end{aligned} \tag{6.1 }$$

and all other $\eta_j^1 = 0$. See (3.10).

Inserting these values for the corresponding η_j in (6.7) we see that all constraints are satisfied with

$$\frac{3}{2}\eta_1 + \frac{\eta_o}{2} = \frac{3}{2}\cdot\frac{6}{7} + \frac{1}{7}\cdot\frac{1}{2} = \frac{19}{14},$$

the same as the value of u^*, thereby confirming optimality. In fact, as our theory [ ] prescribes, we need merely apply the expressions

$$\eta_o = \frac{1}{7}, \quad \xi_1 = \eta_1/\eta_o = \frac{6}{7} \div \frac{1}{7} = 6$$

with all other $\eta_j = 0$ and then reverse the development from (6.6) to (6.7) in order to verify that this value is also optimal for

$$\max R(\xi) = \frac{\frac{3}{2} \cdot 6 + \frac{1}{2}}{6 + 1} = \frac{19}{14}$$

with

(6.13)

$$0 \leq \xi_1 = 6 \leq \frac{2 - (-1)}{g_1} = 6$$

$$0 \leq \xi_2 = 0 \leq \frac{5 - 1}{g_2} = 4$$

$$0 \leq \xi_3 = 0 \leq \frac{1 - 0}{g_3} = 2$$

Evidently we can now directly effect substitutions in (6.1) and obtain

(6.14)

$$y_1 = g_1 \xi_1 + a_1 = 3 - 1 = 2$$

$$y_2 = g_2 \xi_2 + a_2 = 0 + 1 = 1$$

$$y_3 = g_3 \xi_3 + a_3 = 0 + 0 = 0$$

Then we can proceed exactly as in (5.9) to obtain the values $x_1 = 2$, $x_2 = 1$, $x_3 = 0$ which we previously observed to be optimal.

Conclusion:

Before proceeding any further we should probably point up, again, the crucial role played by the general theory (including the transformations and proof procedures) which we introduced in [8] for making explicit contacts between linear fractional and ordinary linear programming--in all generality and exact detail. These transformations are also utilized in the present paper and the theory is also extended via the duality (and other) characterizations given in the preceding text. These are joined together here for the proofs in algorithmic format kind we have just illustrated by example and commentary. Other uses can undoubtedly also be made of this theory and the preceding extensions via the passage (up and back) between linear fractional and ordinary linear programming that is now possible.

Our general theory has been used by others, too, to extend or simplify parts of linear fractional programming en route to effecting the contacts with ordinary linear programming that are thereby obtained. The work of Zionts [22] should perhaps be singled out as being most immediately in line with the $R = \infty$ results presented in this paper. Zionts' development is directed only toward simplifying matters by focusing on eliminating cases for linear fractional programming which are either deemed to be unwanted or of little interest for practical applications.

The developments cease as soon as the contacts with ordinary linear programming are identified via our theory, which he like others utilizes for this purpose.

We have effected the developments in this paper in a way that makes contact with interval linear programming.[1] An opening for further two-way

[1] See [2], [3], [4].

flows is thereby also provided. The resulting junctures should also help to guide subsequent developments in the more special situations that now seem to invite consideration in the future. Finally, the possibilities for dealing with specially structured problems (such as those observed at the start of the present paper) should also be observed explicitly in this conclusion, partly because the theory we have now developed and presented should also be a helpful guide to these additional cases which are important in their own right. Thus we can now conclude here by referring back to our opening remarks.

ADDENDUM

Although the development in this paper proceeded, for clarity, by algorithmic format, we summarize below the explicit solution in tabular format for direct theoretical interpretation and utilization.

Summary of Solution

Denominator	Transformed Numerator	R^*
Bisignant		∞
Unisignant		
(a) positive		$u^s = \dfrac{\gamma_o + \sum_1^s \gamma_2 \delta_2}{1 + \sum_1^s \delta_2}$, least s with $u^s \geq \gamma_{s+1}$
(b) non-negative	$\gamma_o > 0$	∞
	$\gamma_o = 0$	γ_1
	$\gamma_o < 0$	$u^s = \dfrac{\gamma_o + \sum_1^s \gamma_2 \delta_2}{\sum_1^s \delta_2}$, least with $u^s \geq \gamma_{s+1}$

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